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A TURBULENT BURST MODEL FOR BOUNDARY LAYER FLOWS WITH PRESSURE GRADIENT

L. C. Thomas[1,2] and D. J. Benton[1]

1. University of Tennessee, Knoxville, Tennessee, USA
2. University of Petroleum and Minerals, Dhahran, Saudi Arabia

ABSTRACT

The object of this paper is to develop a surface renewal model of the turbulent burst phenomenon for momentum and energy transfer in the wall region for turbulent boundary layer flows with pressure gradient. In addition to obtaining inner laws for the distributions in velocity and temperature, predictions are obtained for the effect of pressure gradient on the mean burst frequency and on the turbulent Prandtl number within the wall region for slight favorable and mild adverse pressure gradients.

1. INTRODUCTION

Based on experimental studies of heat transfer in turbulent boundary layer flow of air, three conclusions have been put forth in the literature. First, the turbulent Prandtl number Pr_t for boundary layer flow of air is of the order of unity across the entire inner region, such that the simple Reynolds analogy ($Pr_t = 1$) can be used as a first approximation. Second, Pr_t apparently falls from a value higher than unity very close to the wall to a nearly constant value of less than unity in the turbulent core. In this connection, the situation within the wall region has been judged to be very vexing "because it is extremely difficult to make accurate measurements in this region and yet it seems evident that something interesting and important is happening in the range of y^+ from 10 to 15" (1). Third, there appears to be a small effect of pressure gradient. What was left undecided in the Stanford studies is whether the high values of Pr_t noted in the viscous sublayer for air are the result of molecular conduction losses during the flight of eddies across the boundary layer or result from some other aspect of the mechanism of the sublayer bursting phenomenon (2).

The damping factor approach to modeling fluid flow in the wall region was introduced in 1956 by van Driest (3). For the development of this approach, van Driest reasoned that the damping effect of a stationary wall on a turbulent fluctuating flow field should be similar to the effect of a plate which oscillates in its own plane in a stationary fluid. The momentum and initial-boundary conditions for this Stokes type flow were written as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

$u(o,y) = 0$, $u(t,o) = u_o \cos(\omega t)$, and $u(t,\infty) = 0$, where u_o and ω represent the amplitude and the frequency of the plate oscillation, and t is the process time. The solution to this system of equation is

$$u = u_o \exp\left(-\frac{y}{a}\right) \cos\left(\omega t - \frac{y}{a}\right) \tag{2}$$

where $a = \sqrt{2\nu/\omega}$. By dropping the unsteady cosine term, van Driest concluded that the maximum amplitude of the motion diminishes with distance from the plate by the factor exp $(-y/a)$. van Driest then reasoned that when the plate is fixed and the fluid oscillates relative to the plate, the factor $[1 - \exp(-y/a)]$ must be applied to the fluid oscillation to obtain the damping effect of the wall. Assuming that $\ell = \kappa y$ in the part of the inner region which is sufficiently remote from the wall that the eddy motion is not damped, van Driest proposed that the mixing length be approximated by

$$\ell = \kappa y\left[1 - \exp\left(-\frac{y}{a}\right)\right] \tag{3}$$

in the wall region. Introducing the dimensionless wall parameters, this equation takes the form

$$\ell^+ = \kappa y^+ \left[1 - \exp\left(-\frac{y^+}{a^+}\right)\right] \tag{4}$$

where $a^+ = \sqrt{2}U^*/\sqrt{\nu\omega}$. With ℓ^+ given by Equation (4), the dimensionless mean velocity distribution within the inner region is represented by

$$u^+ = \frac{\bar{u}}{U^*} = 2\int_0^{y^+} \frac{\bar{\tau}/\bar{\tau}_o \; dy^+}{1+\sqrt{1+4(\kappa y^+)^2[1-\exp(-y^+/a^+)]^2\bar{\tau}/\bar{\tau}_o}} \tag{5}$$

where $U^* = \sqrt{\bar{\tau}_o/\rho}$; $\bar{\tau}/\bar{\tau}_o = 1$ for uniform free-stream velocity. Based on a comparison between mean velocity profiles computed with the aid of Equation (5) (with κ set equal to 0.4) and experimental data for fully turbulent flow in a pipe, van Driest empirically established a value for a^+ of 26.

This approach has been extended to boundary layer flows with pressure gradient by Kays et al.

(2,4-6) and others. In the approach by Kays and co-workers, $\overline{\tau}/\tau_o$ is approximated by the Couette flow equation,

$$\frac{\overline{\tau}}{\tau_o} = 1 + P^+y^+ \quad (6)$$

for mild favorable and moderate adverse pressure gradient flows and a^+ is taken as a function of P^+. Based on a comparison of Equation (5) with experimental data, the Stanford group recommended a correlation for a^+ of the form

$$a^+ = \frac{25}{1 + 7.1\ bP^+} \quad (7)$$

where $b = 2.9$ for $P^+ > 0$, $b = 4.25$ for $P^+ < 0$; $P^+ = (\nu d\overline{P}/dx)/(\rho U^{*3})$.

Attempts have also been made to develop damping factor type analyses for energy transfer in the wall region. In this approach, the unsteady one-dimensional (t,y) energy equation is solved for the situation in which the surface temperature of a plate oscillates in a thermally stationary fluid. The damping effect of the wall on ε_H is then inferred by discarding the unsteady terms in the solution for the instantaneous temperature distribution and by the use of several simplifying assumptions. However, the resulting damping factor formulations for eddy thermal diffusivity ε_H (or turbulent Prandtl number Pr_t) appear to have inconsistencies and have not been widely used in analytical and numerical methods for analyzing turbulent thermal boundary layers. Instead, Pr_t is generally simply set equal to a constant of the order of unity for fluids with moderate to high values of Prandtl number Pr. The fact that an artificial wall/fluid perspective is employed appears to restrict the generality and usefulness of the damping factor approach, especially in heat transfer applications. In general, because the major concern of turbulence model developers has been with the hydrodynamics, relatively little attention has been given to those aspects of turbulence modeling which concerns heat transfer directly (7).

Another approach to analyzing wall turbulence has evolved over the past few years which involves modeling the turbulent fluid flow and energy transfer mechanisms within the large scale coherent structures which are associated with the turbulent burst phenomenon. Like the damping factor approach, this surface renewal model treats wall turbulence as an unsteady transport phenomenon. However, in this approach to modeling turbulent transport within the wall region, the fluid is taken as the fluctuating or intermittent medium with the relative velocity and temperature of the fluid at the wall appropriately set equal to zero, and the contribution of the unsteady fluctuating velocity and temperature distributions to the mean profiles are accounted for statistically. Furthermore, this approach does not involve the use of mixing length or turbulent Prandtl number inputs within the wall region. As pointed out by Reynolds (8), the fundamental advantage of the surface renewal approach is its relation to a specific and not unrealistic picture of events in the viscous sublayer.

The object of this paper is to develop a surface renewal model of the turbulent burst phenomenon for momentum and energy trnasfer in the wall region for turbulent boundary layer flows with pressure gradient. In addition to obtaining inner laws for the distributions in dimensionless velocity u^+ and temperature T^+, predictions are obtained for the effect of pressure gradient P^+ on the mean burst frequency s^+ and on the turbulent Prandtl number Pr_t within the wall region. Because of its prominance in the literature and because of its proven usefulness in characterizing the mean dimensionless velocity distribution in the inner region for fully turbulent boundary layer flows with pressure gradient, the van Driest damping factor approach will be used as a standard for comparison for the fluid flow aspects.

2. ANALYSIS

In the present analysis, the surface renewal model is used in the wall region and a classical eddy diffusivity/mixing length approach is used for the intermediate (overlap) region.

2.1 Wall Region

The instantaneous momentum and energy transfer occuring within a coherent large scale structure during the period of time between inrush and ejection are approximated by

$$\frac{\partial u}{\partial \theta} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\partial P}{\partial x} \quad (8)$$

$$\frac{\partial T}{\partial \theta} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (9)$$

where θ is the age of the event, $u(0,y) = U_i$, $u(\theta,0) = 0$, $u(\theta,y_M) = u_M$, $T(0,y) = T_i$, $T(\theta,0) = T_o$, and $T(\theta,y_H) = T_H$; U_i and T_i represent the distributions in velocity and temperature at the instant of inrush, and u_M and T_H represent the conditions at the interface between the wall region and turbulent core.

The equations are transformed into the mean domain by introducing the age distribution concept; that is,

$$\overline{u} = \int_0^\infty u\phi(\theta,\overline{s})d\theta, \qquad \overline{T} = \int_0^\infty T\phi(\theta,\overline{s})d\theta \quad (10,11)$$

where the age distribution function $\phi(\theta,\overline{s})$ is defined such that the product $\phi(\theta,\overline{s})d\theta$ represents the fraction of the coherent structures residing within the wall region with age between θ and $\theta+d\theta$. The use of these equations and Danckwerts' (9) exponential age distribution,

$$\phi(\theta,\overline{s}) = \overline{s}\exp(-\theta\overline{s}) \quad (12)$$

to transform Equations (8) and (9) and accompanying initial and boundary conditions gives

$$\overline{s}(\overline{u} - U_i) = \nu \frac{d^2\overline{u}}{dy^2} - \frac{1}{\rho}\frac{d\overline{P}}{dx} \quad (13)$$

$$\bar{s}(\bar{T} - T_i) = \alpha \frac{d^2\bar{T}}{dy^2} \quad (14)$$

and $\bar{u}(o) = 0$, $u(y_M) = \bar{u}_M$, $\bar{T}(o) = \bar{T}_o$, and $\bar{T}(y_H) = \bar{T}_H$.

The solution to this system of equations is given in terms of u^+ and T^+ by

$$u^+ = \{u_M^+ \sinh(y^+\sqrt{s^+}) + (U_i^+ - \frac{P^+}{s^+})[\sinh(y_M^+\sqrt{s^+}) - \sinh(y^+\sqrt{s^+}) - \sinh((y_M^+ - y^+)\sqrt{s^+})]\} \frac{1}{\sinh(y^+\sqrt{s^+})} \quad (15)$$

and

$$T^+ = \frac{\bar{T}_o - \bar{T}}{\bar{q}''_o} \rho c_p U^* = \{T_H^+ \sinh(y^+\sqrt{s^+ Pr}) + T_i^+[\sinh(y_H^+\sqrt{s^+ Pr}) - \sinh(y^+\sqrt{s^+ Pr}) - \sinh((y_H^+ - y^+)\sqrt{s^+ Pr})]\} \frac{1}{\sinh(y_H^+\sqrt{s^+ Pr})} \quad (16)$$

To close the analysis, the modeling parameters s^+, U_i^+, u_M^+, T_i^+, T_H^+, y_M^+, and y_H^+ must be specified. This step is taken in a later section.

2.2 Intermediate Region

Analytical solutions can be obtained for u^+ and T^+ in the inner part of the turbulent core on the basis of the classical approach by neglecting the molecular viscous and conduction components of the apparent turbulent stress and heat flux and by employing Couette flow approximations.

Taking these steps, an expression is obtained for u^+ as follows:

$$u^+ = \int \frac{\overline{\tau/\tau_o}}{\kappa y^+} dy^+ + C = \frac{1}{\kappa} \int_{y_j^+}^{y^+} \frac{\sqrt{1+P^+y^+}}{y^+} dy^+ + u_j^+$$

$$= \frac{1}{\kappa} [2(\sqrt{1+P^+y^+} - 1) + \ln(\frac{4}{P^+} \frac{\sqrt{1+P^+y^+} - 1}{\sqrt{1+P^+y^+} + 1})] + C \quad (17)$$

where

$$C = u_j^+ + \frac{1}{\kappa} [2(1 - \sqrt{1+P^+y_j^+}) + \ln \frac{(\sqrt{1+P^+y_j^+} + 1)^2}{4y_j^+}] \quad (18)$$

Similarly, an expression is obtained for T^+ by writing

$$T^+ = \int \frac{Pr_t}{\kappa y^+} \frac{\overline{q''}/\overline{q''_o}}{\sqrt{\overline{\tau}/\overline{\tau}_o}} dy^+ + B \quad (19)$$

Setting Pr_t equal to $Pr_{t\infty}$ in the turbulent core and using Couette flow approximations, this equation becomes

$$T^+ = \frac{Pr_{t\infty}}{\kappa} \int_{y_j^+}^{y^+} \frac{dy^+}{y^+\sqrt{1+P^+y^+}} + T_j^+ \quad (20)$$

such that

$$T^+ = \frac{Pr_{t\infty}}{\kappa} \ln(\frac{4}{P^+} \frac{\sqrt{1+P^+y^+} - 1}{\sqrt{1+P^+y^+} + 1}) + B \quad (21)$$

where

$$B = T_j^+ + \frac{Pr_{t\infty}}{\kappa} \ln \frac{(\sqrt{1+P^+y_j^+} + 1)^2}{4y_j^+} \quad (22)$$

Because Equations (17)-(22) only apply in the Couette flow region where convective effects are small, the use of these equations for favorable pressure gradients ($P^+ < 0$) is restricted to the close vicinity of the wall in which the term $1 + P^+y^+$ is well above zero. Thus, the fact that the terms involving $1 + P^+y^+$ and $1 + P^+y_j^+$ can be imaginary for favorable pressure gradients is of no practical concern.

Based on experimental data κ is set equal to 0.41 and $Pr_{t\infty}$ is approximated by 0.86. The conditions $u_j^+ = u^+(y_j^+)$ and $T_j^+ = T^+(y_j^+)$ are obtained by interfacing Equations (17) and (21) with equations for u^+ and T^+ that apply to the wall region.

In the limit as P^+ approaches zero, Equations (17) and (21) reduce to the familiar logarithmic intermediate laws

$$u^+ = \frac{1}{\kappa} \ln y^+ + C \quad (23)$$

$$T^+ = \frac{Pr_{t\infty}}{\kappa} \ln y^+ + B \quad (24)$$

2.3 Model Closure

The hydrodynamic parameters U_i^+, u_M^+, and y_M^+ can be expressed in terms of s^+ and the other independent varables by satisfying the Newton law of viscous shear at the wall

$$\frac{du^+}{dy^+} = 1 \qquad \text{at } y^+ = 0 \quad (25)$$

and by requiring continuity in du^+/dy^+ and d^2u^+/dy^{+2} between Equations (15) and (17) at $y_j^+ = y_M^+$. In addition, the parameter C associated with the intermediate region is evaluated by maintaining continuity in u^+. Because of the scarcity of experimental data for mean burst frequency $\bar{s}$ for turbulent boundary layer flows with nonzero pressure gradient, the approach used by Kays and associates (2,4-6) in evaluating the damping parameter a^+ is used. That is, s^+ is determined such that u^+ is in agreement with experimental data in the intermediate region for various values of P^+. Using the damping factor equation, Equation (5), to represent the experimental data in the intermediate region, s^+ and the other hydrodynamic parameters have been calculated by algebraic elimination and numerical iteration. Calculations are shown in Figure 1 for s^+ and C as a function of P^+. The calculations for u^+ obtained on the basis of Equations (15) and (17) are shown in Figure 2. Calculations obtained on the basis of the damping factor approach are also shown.

With s^+ and the other hydrodynamic characteristics known, the thermal parameters T_i^+, T_H^+, y_H^+, and B are evaluated by satisfying the Fourier law of conduction at the wall,

$$\frac{dT^+}{dy^+} = Pr \qquad \text{at } y^+ = 0 \qquad (26)$$

and by maintaining continuity in T^+, dT^+/dy^+ and d^2T^+/dy^{+2} between Equations (16) and (21) at $y_3^+ = y_H^+$. Calculations obtained for B and the distribution in temperature T^+ are shown in Figure 1 and 3 for air (Pr = 0.72). Calculations for T^+ obtained on the basis of the damping factor approach are also shown in Figure 3.

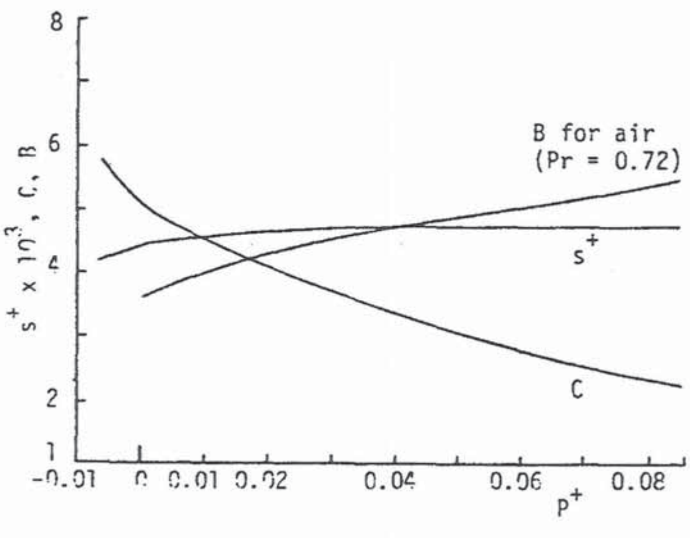

Figure 1. Calculations for s^+, C, and B.

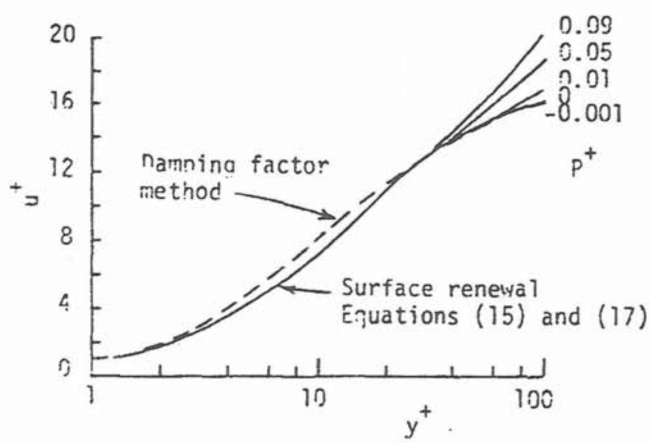

Figure 2. Calculations for u^+ in the inner region.

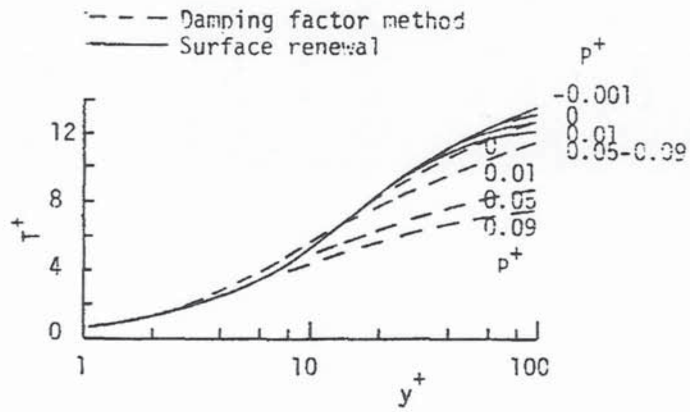

Figure 3. Calculations for T^+ in the inner region for air (Pr = 0.72).

3. DISCUSSION

As indicated in the previous section, the Couette flow approximations employed in both the surface renewal and damping factor analyses apply within the inner region for adverse pressure gradients and for very mild favorable pressure gradients. For moderate to strong favorable pressure gradient flows, convective effects are significant within this important region. Therefore, we will restrict our attention to adverse ($P^+ > 0$) and very mild favorable ($P^+ \gtrsim -0.001$) pressure gradient flows for which the Couette flow approximations are valid throughout the entire inner region.

Referring to Figure 2, the calculations for dimensionless velocity u^+ within the inner region obtained on the basis of the present surface renewal formulation and the traditional damping factor approach are seen to be in basic agreement. Although P^+ has a noticable effect on the calculations for u^+ in the intermediate region, the calculations for u^+ within the wall region are essentially insensitive to changes in P^+ in the range -0.001 to +0.01. The surface renewal and damping factor relationships for u^+ are compared with experimental data for P^+ equal to 0 and 0.01 in Figure 4. The small difference in calculations for u^+ in the region $y^+ \gtrsim 30$ is within the data scatter.

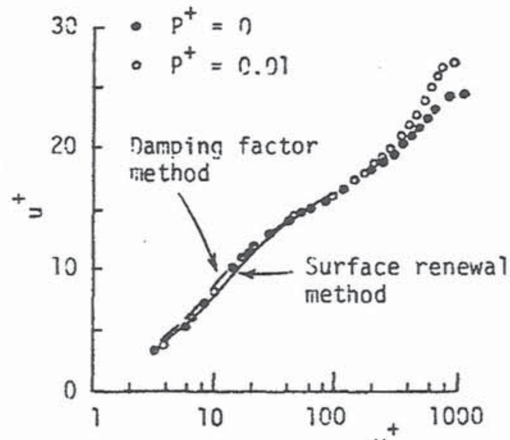

Figure 4. Comparison of predictions for u^+ with experimental data by Kays and Moffat(1).

The surface renewal calculations for dimensionless temperature distribution T^+ shown in Figure 3 are even less sensitive to changes in P^+ than the calculations for u^+. On the other hand, calculations obtained on the basis of the damping factor approach (with Pr_t specified by the equation recommended by Kays and Crawford (2) - See Figure 6a) are quite sensitive to P^+. (Calculations obtained by the use of $Pr_t = 1$ are essentially the same.) Surface renewal and damping factor calculations for T^+ are compared with representative data in Figure 5 for a mild adverse pressure gradient flow. Whereas the surface renewal calculations are in excellent agreement with these data, the damping factor calculations lie about 16% too low. In general, the profiles of T^+ obtained by Kays and Moffat (1) were reported to be less effected by deceleration than the profiles for u^+, which is consistent with the present analysis

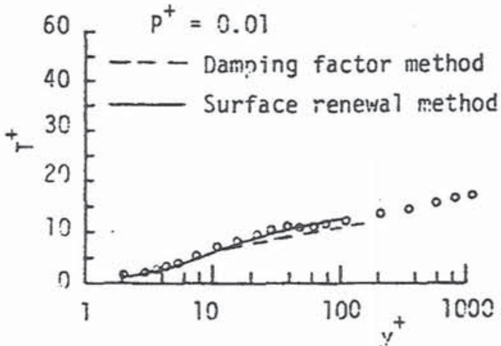

Figure 5. Comparison of predictions for T^+ with experimental data for air (Pr = 0.72)

As indicated earlier, one of the benefits of the surface renewal approach is that inputs are not required for turbulent Prandtl number Pr_t within the wall region. It follows that the surface renewal approach provides a theoretical basis for developing predictions for Pr_t in the important wall region. To calculate Pr_t, we write

$$Pr_t = \frac{\dfrac{\overline{\tau/\tau_o}}{du^+/dy^+} - 1}{\dfrac{\overline{q''/q''_o}}{dT^+/dy^+} - \dfrac{1}{Pr}} \qquad (27)$$

where the Couette flow approximations are used for $\overline{\tau/\tau_o}$ and $\overline{q/q''_o}$ (i.e., Equation (6) and $\overline{q''/q''_o} = 1$). Using Equations (15) and (16) for u^+ and T^+ within the wall region, calculations for Pr_t are compared with experimental data in Figures 6 a and 6b for both zero and mild adverse pressure gradient flows. Based on this analysis, the effect of an adverse pressure gradient is to increase Pr_t in the wall region, with a maximum occuring in the vicinity of $y^+ = 40$. Considering the uncertainties in experimental measurements for Pr_t within the wall region, the surface renewal analysis is in reasonably good agreement with the data.

Referring to Figure 1, the calculations for s^+ increase very slightly with P^+. As has previously been shown (10,11), the calculations for s^+ are compatible with experimental measurements of mean burst frequency for fully turbulent flows with very small favorable and zero pressure gradients. Because U^* decreases with increasing values of P^+, the predicted mean burst frequency $\overline{s}$ actually decreases with increasing P^+. To expand on this point, predictions for $\overline{s}$ obtained on the basis of the present analysis are compared in Figure 7 with experimental data of Strickland and Simpson (12) for a strongly nonequilibrium adverse pressure gradient flow. Beacuse the surface renewal analysis was developed for near equilibrium conditions, the calculations for s^+ generally lie well above the data. But the predicted effect of P^+ on $\overline{s}$ is consistent with the data.

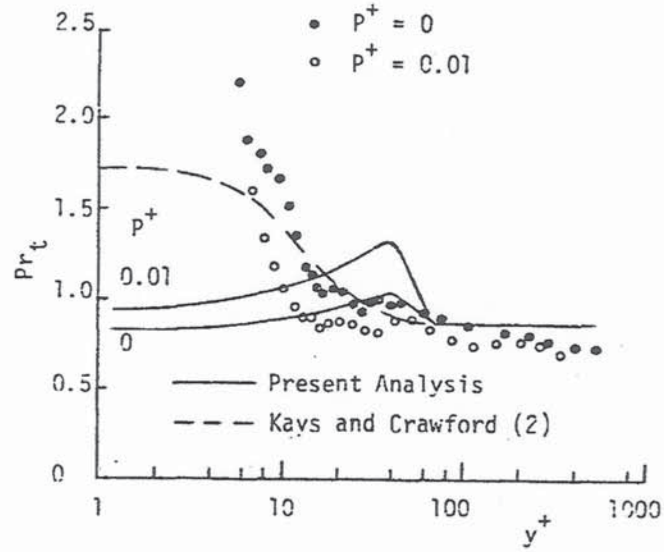

Figure 6a. Comparison of predictions for Pr_t with experimental data by Blackwell (6).

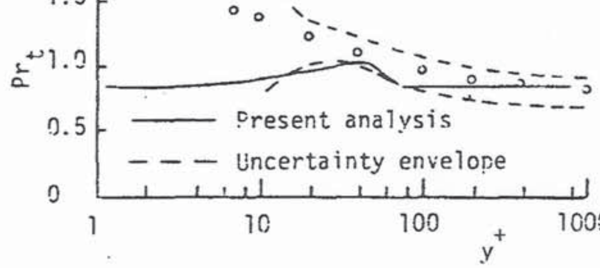

Figure 6b. Comparison of predictions for Pr_t with experimental data by Strickland and Simpson (12).

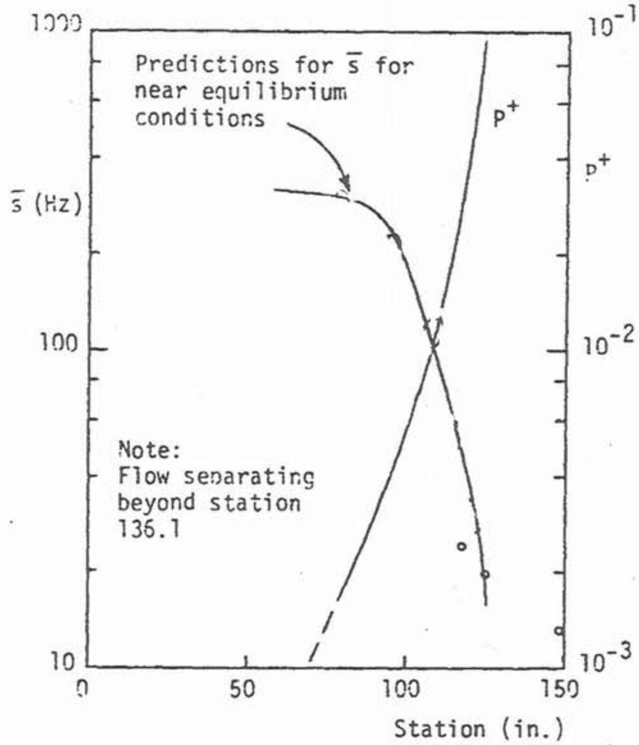

Figure 7. Mean burst frequencies for adverse pressure gradient flow; experimental data by Strickland and Simpson (12).

4. CONCLUSIONS

A surface renewal model has been developed in this paper for fluid flow and energy transfer associated with the turbulent burst phenomenon for turbulent boundary layer flows with slight favorable and mild adverse pressure gradients. The analysis indicates that P^+ has little effect on u^+ and T^+ in the innermost part of the wall region ($y^+ \gtrsim 30$). In the outer part of the wall region and intermediate region of a decelerating turbulent flow field, u^+ increases and T^+ decreases with increasing P^+, with T^+ being much less sensitive to changes in P^+ than u^+.

Predictions for the turbulent Prandtl number within the wall region for air were found to increase with increasing P^+, with a maximum occuring at $y^+ \approx 40$. This predicted increase in Pr_t with P^+ is in basic agreement with experimental measurements within the wall region. Based on this analysis, it would appear that the higher values of Pr_t measured in the important region $10 \lesssim y^+ \lesssim 60$ are primarily the result of unsteady transport associated with the turbulent burst mechanism within the wall region.

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