COMPARISON OF METHODS FOR NUMERICAL INTEGRATION IN COMPUTING COOLING TOWER DEMAND CURVES

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ABSTRACT

A comparison is made of seventy-five cases representing seven major methods used for numerical integration as to their respective accuracy, speed, cost-effectiveness, and stability. Gauss quadrature is shown to be by far the preferable method for computing cooling tower demand curves.

NOMENCLATURE

- a = the interfacial area per unit volume as in Equation 3
- C_{pw} = constant pressure specific heat of water
- $F(X)$ = a general function of X
- h = enthalpy of air
- K = mass transfer coefficient
- L = water flux
- N = number of quadrature points as in Equation 1
- T_W = water temperature
- V = volume
- W_I = quadrature weights as in Equation 2
- X = a general independent variable
- X_I = quadrature abscissas as in Equation 2

Subscripts

- a = the lower limit of integration as in Equation 1
- A = referenced to air
- b = the upper limit of integration as in Equation 1
- W = referenced to water

INTRODUCTION

Many engineering problems, and specifically the computing of cooling tower demand curves, require the solution of integral equations. Those which are not analytically integrable require numerical means to resolve. The computation of cooling tower demand curves requires the solution of integral equations such as described by Merkel which involve thermodynamic properties and are not analytically integrable.

Several numerical methods have been applied to this problem. Merkel used the 4-point Chebyshev for its simplicity and ease in hand calculations. LeFevre has pointed out several inaccuracies in Merkel's method, in particular, the use of the 4-point Chebyshev integration. With the introduction of computers, other more accurate methods of integration can be applied as LeFevre suggests. Personal computers make it possible for the engineer to generate custom demand curves. Generating these demand curves requires accurate and timely solution of integral equations.

BACKGROUND

Numerical integration is often called quadrature because of the geometric analogy of finding the area under a curve. Some authors use the terms *numerical integration* and *quadrature* interchangeably while others make a distinction in their use as to whether the function to be integrated is tabular (such as experimental data) or analytical (such as ***SIN(X)***) (Hildebrand p. 381).

Discretization

The basic assumption of numerical integration is that the integral can be approximated by the following discrete formulation (Hildebrand p. 385):

$$\int_a^b F(X)\,dX \approx (b-a)\sum_{I=1}^{N} W_I\, F(X_I) \qquad (1)$$

where W_I are coefficients called *weights*, X_I are specific locations on X called *abscissas*, and N is the number of terms or points.

Some discretizations like Equation 1 also use derivatives of $F(X)$. Because of the involvement of thermodynamic properties in the cooling tower demand integral, using derivatives of $F(X)$ in Equation 1 is not practical. Therefore, the methods compared here include only those methods which do not require the derivatives of $F(X)$.

Weights and Abscissas

Among the methods described by Equation 1, there is the further distinction as to whether the weights, W_I, and abscissas, X_I, are constrained or free. In addition, some criterion is imposed in order to determine either the weights, the abscissas, or both. The two most common criteria are that the method conform to some geometrical analog (e.g., the trapezoidal rule, Hildebrand p. 95) or that it exactly integrate some analytical function or family of functions (e.g., Newton-Cotes will exactly integrate any polynomial up to some order depending on the number of points, Hildebrand p. 93).

Convergence and Stability

A method is said to converge if the series formed by integrations of successively higher order converges as the number of points, $N \rightarrow \infty$. In a practical sense, a method is said to converge if results obtained by successively higher orders asymptotically approaches a constant value. Just because a method converges does not insure that it will converge to the correct answer--that involves the additional consideration of accuracy.

A minimal condition for stability of a method based on Equation 1 is that the weights be positive, $W_i > 0$ (Hildebrand p. 95). The sum of the weights must be equal to 1, $\Sigma W_i = 1$, else the method would not even integrate the case where $F(X) = constant$ correctly. Therefore, if any of the weights are negative, the sum of the absolute value of the weights will exceed 1. If this condition exists, it can be shown that the method will not converge (Hildebrand, p. 96).

Degrees of Freedom

The extent to which a particular method can meet a specific criterion (such as exactly integrating any polynomial up to some order) is limited by the number of degrees of freedom that it has. Or put another way, a method is limited by the number of parameters that can be adjusted in order to meet the criterion.

If the abscissas, X_I, are selected beforehand so as to take on convenient values (equally-spaced intervals yields the Newton-Cotes method), the number of degrees of freedom is limited to N (W_1 through W_N). If the weights, W_I, are selected beforehand (requiring that they all be equal yields the Chebyshev method), the number of degrees of freedom is limited to N (X_1 through X_N). If no such constraint is imposed and optimal values are selected for W_I and X_I (this yields the Gaussian method), the number of degrees of freedom is limited to $2N$ (W_1 through W_N and X_1 through X_N).

METHODS COMPARED

Seven methods were selected for comparison: trapezoidal, Simpson, Newton-Cotes, Romberg, Chebyshev, Lobatto, Gauss, and composite or subdivided Gauss. Other more obscure methods such as Radau, Hermite, Laguerre, Jacobi, and Filon are covered by Hildebrand and Abramowitz and Stegun. Inclusion of these would not add to this comparison as these methods apply to special cases not pertinent to the integral equation being considered.

The Trapezoidal Rule

The trapezoidal rule is based on the geometrical interpretation of integration as illustrated in Figure 1. It seems intuitive that if enough points are taken, the area of the trapezoids will sum to the total area. It can also be seen from the figure that if the function being integrated is consistently cupped upward or downward over the interval, there is always a small error. Thus it will require an infinite number of points to obtain the exact answer. It also follows logically that for any function other than a straight line, there is no guarantee that the trapezoidal rule will ever produce the exact answer in a finite number of points.

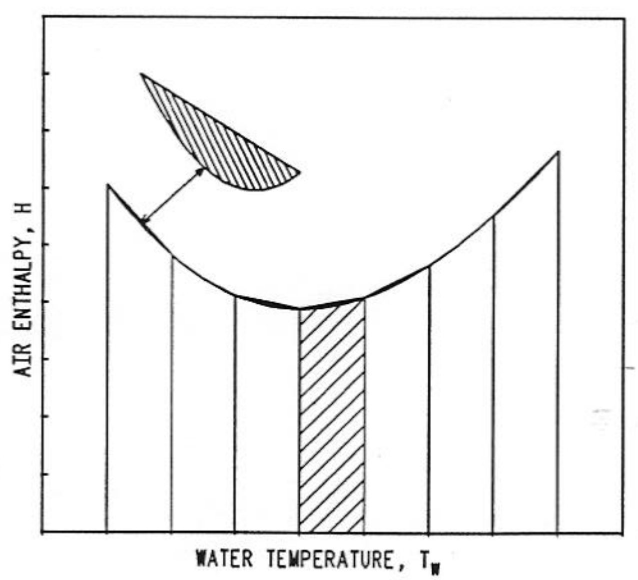

Figure 1. Trapezoidal Rule

Simpson's Method

Simpson's method is very similar to the trapezoidal rule. The difference is that a parabola rather than a straight line is drawn between the points. This is actually quite an improvement; but the same problem exists, albeit at a higher order.

Newton-Cotes Methods

Unlike the trapezoidal and Simpson methods, the Newton-Cotes methods require a different set of weights depending on the number of points. These are found through a complicated process described by Hildebrand and listed in Abramowitz and Stegun. As indicated previously, the Newton-Cotes weights are found by constraining the abscissas, X_I, to be at equally-spaced intervals and requiring that they exactly integrate any polynomial of degree less than N. It is important to note that not all of the weights are positive for $N=9$ and $N>10$. The sum of the absolute value of the weights becomes unbounded as $N \rightarrow \infty$. The Newton-Cotes method is unstable for these cases and, of course, does not converge.

Romberg Integration

Romberg devised a hybrid method whereby Richardson's extrapolation is used to predict the asymptotic result of successively smaller interval trapezoidal integration (Hildebrand p. 99). Richardson's extrapolation could also be used in conjunction with other integration methods; however, these would not be as easily implemented as the trapezoidal rule. Romberg noticed when applying the trapezoidal rule, that the results from the previous integral can be reused to compute the next integral in succession, effectively reducing the work by a factor of two. The Romberg method has the further advantage of providing an estimate of the error as part of the Richardson's extrapolation.

Chebyshev Quadrature

As indicated previously, the Chebyshev method arises from constraining the weights, W_I, to be equal (this assures stability, unlike Newton-Cotes),and seeking the abscissas, X_I, such that the exact integral will be given for any polynomial of order less than N. A different set of abscissas are needed for each value of N. These are found through a complicated process described by Hildebrand and listed in Abramowitz and Stegun.

Lobatto Quadrature

Lobatto quadrature arises from constraining only two of the abscissas, one at each end point (**a** and **b** in Equation 1, which can easily be normalized to -1 and 1). The remaining abscissas and all of the weights are determined optimally such that the exact integral will be given for any polynomial of order less than $2N-2$. A different set of abscissas and weights are needed for each value of N. These are found through a complicated process described by Hildebrand and listed in Abramowitz and Stegun.

Gauss Quadrature

As indicated previously, the Gauss method arises from constraining neither the abscissas or weights. These are found by requiring that the exact integral will be given for any polynomial of order less than $2N$. A different set of abscissas and weights are needed for each value of N. These are found through a complicated process described by Hildebrand and listed in Abramowitz and Stegun. By not constraining the weights to be equal (as with the Chebyshev method) there is no assurance from the outset that all of the weights will be positive. As it turns out, however, they are. Furthermore, the method is stable, convergent, and accurate.

Composite Gauss Quadrature

Composite or subdivided methods are frequently used to integrate to convergence by subdividing until the change in the result for successive subdivisions is less than some tolerance. In this sense these are similar to Romberg's method. Composite methods have also been used as a substitute for higher order methods (i.e., 5-point Gauss quadrature taken over two half intervals as a substitute for 10-point taken over the entire interval). The former use is certainly legitimate; whereas the latter is questionable. These methods are included in the comparison in order to illustrate their diminishing return.

THE TEST CASE

The integral equation from which counterflow cooling tower demand curves are generated was introduced by Merkel:

$$\frac{KaV}{L} = \int C_{pw} \frac{dT_W}{(h_w - h_A)} \quad (2)$$

where K is the mass transfer coefficient, a is the interfacial surface area per unit volume, V is the volume, L is the water flux, C_{pw} is the constant pressure specific heat of the water, h_W is the enthalpy of air at the conditions of the air water interface, h_A is the enthalpy of the air, and T_W is the water temperature.

Important improvements in the accuracy of this equation have been made by others (e.g., LeFevre).

However, the mathematical functionality of the improved forms remain basically the same.

The graphical representation of this integral equation is illustrated in Figure 2. If the term *(h_W-h_A)* were in the numerator of Equation 2 instead of the denominator, the integral would simply be the area between the two process lines (or the shaded area in the figure). If this were the case, the integral would be analogous to ***F(X)dX***.

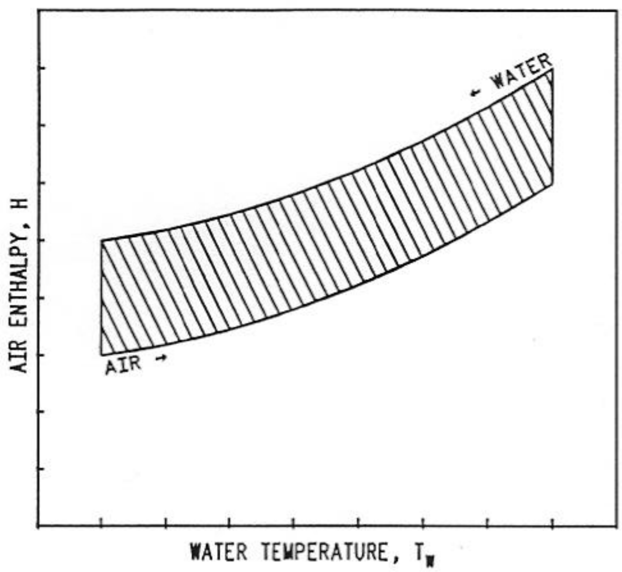

Figure 2. Process Lines

Because the term *(h_W-h_A)* is in the denominator, the integration is more complicated. The integral is more analogous to ***dX/F(X)***. At this point an electrical analogy is often helpful. The mass transfer coefficient is analogous to a conductance. Each infinitesimal element of the interacting volume, ***dV***, along the process line has a particular conductance. These conductances must be added in series. If any one is zero, then the total is zero. Put another way, if there is a break in the circuit (conductance equal to zero), then the entire circuit is broken regardless of the other parts.

A near zero conductance is called a *pinch* because it shows up graphically as a point where the two process lines come very close. The Second Law of Thermodynamics states that the two lines can never touch or cross (provided that the only processes occurring are 2-species heat and mass transfer). A pinch is illustrated in Figure 3. A pinch situation is the most difficult to accurately integrate numerically. Therefore, the test case will focus on this aspect.

Units are unimportant in considering methods for numerical integration as these can be normalized by

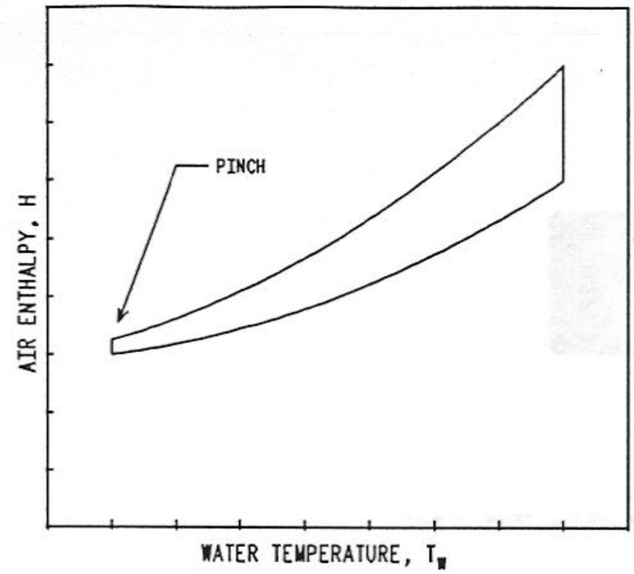

Figure 3. Pinch Point

taking a constant outside of the integral. The simplest analog to the pinch illustrated in Figure 3 is the integral of ***1/X***. The degree of pinch is the maximum distance between the two process lines divided by the minimum. This is typically no more than 3. The most extreme case for which an actual tower design was found slightly exceeded 7. A value of 10 was selected for the test case in order to a have sufficiently stringent test. The test integral is given by Equation 3.

$$\int_1^{10} \frac{dX}{X} = \ln(10) \qquad (3)$$

Which is the same as the integral from 0.1 to 1 or from 10 to 100, etc.

RESULTS

The results of the comparison are given in Tables 1 and 2. The only difference between the two is the order in which the entries are sorted. The first column gives the name of the method and the number of points. The second is the resulting approximation to the integral (Equation 2). The third column is the difference between this and the exact solution (Equation 3). The fourth column is the computer time in seconds (only the relative time required for the various methods is important). The fifth and last column is the number of decimal digits of accuracy achieved per second of computations (again, only the relative measure is important).

DISCUSSION

The accuracy of the various methods (at least for this problem) can be seen in column three of the tables. There is a significant disparity in the accuracy of the various methods (a range of 16 orders of magnitude or 16 digits). For the most part, methods using more points are more accurate than those using fewer points; but there are examples where the same number of points achieves 12 digit different accuracy (100-point trapezoidal vs. 96-point Gauss).

The accuracy of the Chebyshev method surpasses that of the Newton-Cotes for the same number of points even though both methods have the same number of degrees of freedom. The accuracy of the Chebyshev, Lobatto, and Gauss methods (all having optimally-spaced abscissas) significantly exceeds that of the trapezoidal, Simpson, and Newton-Cotes (all having equally-spaced abscissas). These two comparisons illustrate the general rule that freedom in the abscissas is typically more important than freedom in the weights. They also illustrate the general rule that accuracy greatly increases with increasing degrees of freedom.

The accuracy of the trapezoidal rule compares quite poorly with all of the other methods and is of little practical value. Simpson's Rule is a considerable improvement over the trapezoidal rule, but also compares poorly with other methods. Newton-Cotes is useful when restricted to fixed points, provided limited accuracy is sufficient and no more than 9 or 10 points are involved. The Romberg method is easily programmed and highly accurate. The Chebyshev method is of questionable utility when compared to Gauss.

All of the methods compared are stable and converge except the Newton-Cotes and Romberg. Newton-Cotes diverges for this example if more than 40 points are used (this can happen with fewer points for a different integral). The reason for this divergence has already been covered. The Romberg method will also diverge. For this case the error is reduced up to 1025 points and then increases. Although it is of questionable practical concern (16385+ point integration is certainly excessive), because the Romberg method employs Richardson's extrapolation, the differences computed using limited precision arithmetic will eventually cause the process to diverge.

It can be seen from the last column in the tables that there is an optimal number of points from the perspective of digits of accuracy per second of computer time for each method (except the composite Gauss). These are approximately 10, 10, 9, 9, 6, 10, and 20-points respectively for the trapezoidal, Simpson, Newton-Cotes, Romberg, Chebyshev, Lobatto, and Gauss methods respectively. The Gauss method has the largest number of points at the optimum. This is because it also has the largest number of degrees of freedom per point.

The accuracy of the methods at this optimum varies considerably (10 digits). What this optimum means is that there is a diminishing return for using any more or less points. If the accuracy at the optimum is not sufficient for the application, then the method is not cost-effective.

The optimum return of the methods also varies significantly. These are approximately 6, 10, 25, 25, 12, 31, 47, and 56 digits per second for the trapezoidal, Simpson, Newton-Cotes, Romberg, Chebyshev, Lobatto, and Gauss methods respectively. The return for the Gauss is considerably higher than for any of the others. Again, this is a result of the number of degrees of freedom per point.

Note that the composite Gauss has a diminishing return for <u>all</u> cases (i.e., no maximum is exhibited). Also note that the accuracy of the 2*5-point Gauss is 2.6 digits less than 10-point Gauss. Using a composite rule as a substitute for a higher order method is never cost-effective and should only be used if the higher order method is unstable (Hildebrand p. 95).

The most significant results from a practical standpoint are revealed by the sorting in Table 2, where the methods are arranged in diminishing return. The top of the table is dominated by optimally-spaced abscissa methods. These are the Gauss, Lobatto, and Chebyshev, in that order. This is also the order of decreasing number of degrees of freedom for the same number of points (recall that the Lobatto fixed the end points and Chebyshev fixed the weights).

It is also interesting to rank the methods as to their first occurrence in Table 2. In first place is 20-point Gauss, followed by 10-point Lobatto in ninth, 6-point Chebyshev 17th, and 11-point Newton-Cotes in 23rd place. These are followed at a considerable distance by 9-point Romberg in 43rd, 5-point Simpson 47th, and 10-point trapezoidal in 55th place.

While the Romberg is the second most accurate method, it is third from the last in cost-effectiveness. This is because it is based on the trapezoidal rule with Richardson's extrapolation to improve the accuracy. Still the number of degrees of freedom per point is small when compared to Gauss. The advantage to the Romberg method is the error estimation, not its computational efficiency.

CONCLUSIONS

For this application Gauss quadrature is significantly more accurate than any other method given the same number of points. The cost-effectiveness of the 20-point Gauss method is considerably greater than any other method. The accuracy of 10-point Gauss or Lobatto quadrature is probably sufficient for computing demand curves with negligible loss of cost-effectiveness. Lobatto quadrature has the advantage of including the process end-points. The Romberg method is by far the most accurate of those having equally-spaced points. Only diminishing return is seen for the composite or subdivided Gauss method.

SUMMARY

A comparison was made of seven major methods used for numerical integration. Based on accuracy, stability, and cost-effectiveness, Gauss or Lobatto quadrature was found to be by far the preferable methods for computing cooling tower demand curves. The 20-point Gauss method had the highest cost-effectiveness. The accuracy of 10-point Gauss or Lobatto quadrature is probably sufficient and is almost as cost-effective while requiring only half of the computational time.

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Table 1. Results by Method

METHOD	RESULT	ERROR	SEC	DIGITS/SEC
exact=ln(10)	2.302585092994046	NA	NA	NA
3-point trapezoidal	3.293181818181818	.990596725187773	.0556	.074
5-point trapezoidal	2.629221182043763	.326636089049717	.0934	5.203
10-point trapezoidal	2.378968253968254	.076383160974208	.1879	5.946
20-point trapezoidal	2.320713727992899	.018128634998853	.3731	4.668
100-point trapezoida	2.303266344275725	.000681251281679	1.8611	1.702
1000-point trapezoid	2.302591788824242	.000006695830197	18.5000	.280
10000-point trapezoi	2.302585159832371	.000000066838326	185.7000	.039
3-point Simpson	2.740909090909091	.438323997915045	.0549	6.523
5-point Simpson	2.407900969997744	.105315877003698	.0932	10.485
10-point Simpson	2.320575735326238	.017990642332192	.1689	10.330
20-point Simpson	2.303996791535492	.001411698541446	.3539	8.054
100-point Simpson	2.302587417884465	.000002324890419	1.8364	3.068
1000-point Simpson	2.302585093214436	.000000000220390	18.5833	.520
10000-point Simpson	2.302585092994053	.000000000000007	185.6000	.076
4-point Newton-Cotes	2.563392857142857	.260807764148811	.0520	11.221
5-point Newton-Cotes	2.385700428603654	.083115335609608	.0629	17.177
6-point Newton-Cotes	2.359816280203757	.057231187209711	.0589	21.089
7-point Newton-Cotes	2.324715568045400	.022130475051354	.0664	24.932
8-point Newton-Cotes	2.318756672902205	.016171579908159	.0780	22.964
9-point Newton-Cotes	2.309464383406782	.006879290412736	.0849	25.476
10-point Newton-Cote	2.307779655612245	.005194562618199	.0970	23.549
11-point Newton-Cote	2.304925264954663	.002340171960618	.1032	25.496
12-point Newton-Cote	2.304389575052643	.001804482058597	.1166	23.540
16-point Newton-Cote	2.302836691040503	.000251598046457	.1496	24.067
20-point Newton-Cote	2.302624627351999	.000039534357953	.1923	22.896
30-point Newton-Cote	2.302585604701296	.000000511707250	.2792	22.535
40-point Newton-Cote	2.302585101885737	.000000008891691	.3846	20.933
50-point Newton-Cote	2.302585386278535	.000000293284489	.4630	14.111
60-point Newton-Cote	2.299587560505099	.002997532488947	.5739	4.397
3-point Romberg	2.740909090909091	.438323997915045	.0563	6.362
5-point Romberg	2.385700428603654	.083115335609608	.0948	11.394
9-point Romberg	2.313627920068950	.011042827074904	.1683	11.625
17-point Romberg	2.303414977334841	.000829884340795	.3155	9.767
33-point Romberg	2.302615169490732	.000030076496687	.6054	7.469
65-point Romberg	2.302585558689706	.000000465695660	1.1882	5.329
129-point Romberg	2.302585095812346	.000000002818300	2.3372	3.658
257-point Romberg	2.302585093000290	.000000000006244	4.6682	2.400
513-point Romberg	2.302585092994049	.000000000000004	9.2909	1.555
1025-point Romberg	2.302585092994044	.000000000000002	18.5000	.797
2049-point Romberg	2.302585092994055	.000000000000009	36.9667	.380
4097-point Romberg	2.302585092994049	.000000000000003	73.8500	.196
8193-point Romberg	2.302585092994024	.000000000000022	148.3000	.092
16385-point Romberg	2.302585092994025	.000000000000020	295.5000	.046
3-point Chebyshev	2.185206098280966	.117378994713080	.0434	21.441
4-point Chebyshev	2.255234587073967	.047350505920079	.0522	25.386
5-point Chebyshev	2.270694358126710	.031890734867336	.0646	23.163
6-point Chebyshev	2.288199209959526	.014385883034520	.0588	31.343
7-point Chebyshev	2.292318127020230	.010266965973816	.0680	29.245
9-point Chebyshev	2.298956393557629	.003628699436417	.0874	27.923
3-point Lobatto	2.740909090909091	.438323997915045	.0430	8.339
4-point Lobatto	2.399427496016199	.096842403022153	.0511	19.825
5-point Lobatto	2.326155917967643	.023570824973597	.0638	25.508
6-point Lobatto	2.308566606914805	.005981513920760	.0581	38.239
7-point Lobatto	2.304135007946037	.001549914951991	.0676	41.544
8-point Lobatto	2.302991462666707	.000406369672661	.0773	43.865
9-point Lobatto	2.302692421630770	.000107328636724	.0868	45.736
10-point Lobatto	2.302613573419385	.000028480425339	.0965	47.091
2-point Gauss	2.106382978723405	.196202114270641	.0327	21.662
3-point Gauss	2.246609743847312	.055975349146733	.0434	28.852
4-point Gauss	2.286969523872802	.015615569121244	.0523	34.565
5-point Gauss	2.298283110737116	.004301982256930	.0646	36.631
6-point Gauss	2.301408084107758	.001177008886287	.0590	49.666
7-point Gauss	2.302264348288730	.000320744705316	.0681	51.313
8-point Gauss	2.302497902032418	.000087190961628	.0781	51.962
9-point Gauss	2.302561429367133	.000023663626913	.0872	53.025
10-point Gauss	2.302578677886270	.000006415107776	.0974	53.323
12-point Gauss	2.302584622579007	.000000470415039	.1163	54.398
16-point Gauss	2.302585090482857	.000000002511189	.1553	55.385
20-point Gauss	2.302585092979036	.000000000015010	.1925	56.218
40-point Gauss	2.302585092995388	.000000000001342	.3855	30.797
96-point Gauss	2.302585092994049	.000000000000004	.9191	15.721
2*5-point Gauss	2.302323045787393	.000262047206653	.0971	36.891
10*5-point Gauss	2.302585083047175	.000000009946871	.4808	16.645
20*5-point Gauss	2.302585092966214	.000000000027832	.9585	11.013
100*5-point Gauss	2.302585092992895	.000000000001151	4.7955	2.490

Table 2. Results by Cost-Effectiveness

METHOD	RESULT	ERROR	SEC	DIGITS/SEC
exact=ln(10)	2.302585092994046	NA	NA	NA
20-point Gauss	2.302585092979036	.000000000015010	.1925	56.218
16-point Gauss	2.302585090482857	.000000002511189	.1553	55.385
12-point Gauss	2.302584622579007	.000000470415039	.1163	54.398
10-point Gauss	2.302578677886270	.000006415107776	.0974	53.323
9-point Gauss	2.302561429367133	.000023663626913	.0872	53.025
8-point Gauss	2.302497902032418	.000087190961628	.0781	51.962
7-point Gauss	2.302264348288730	.000320744705316	.0681	51.313
6-point Gauss	2.301408084107758	.001177008886287	.0590	49.666
10-point Lobatto	2.302613573419385	.000028480425339	.0965	47.091
9-point Lobatto	2.302692421630770	.000107328636724	.0868	45.736
8-point Lobatto	2.302991462666707	.000406369672661	.0773	43.865
7-point Lobatto	2.304135007946037	.001549914951991	.0676	41.544
6-point Lobatto	2.308566606914805	.005981513920760	.0581	38.239
2*5-point Gauss	2.302323045787393	.000262047206653	.0971	36.891
5-point Gauss	2.298283110737116	.004301982256930	.0646	36.631
4-point Gauss	2.286969523872802	.015615569121244	.0523	34.565
6-point Chebyshev	2.288199209959526	.014385883034520	.0588	31.343
40-point Gauss	2.302585092995388	.000000000001342	.3855	30.797
7-point Chebyshev	2.292318127020230	.010266965973816	.0680	29.245
3-point Gauss	2.246609743847312	.055975349146733	.0434	28.852
9-point Chebyshev	2.298956393557629	.003628699436417	.0874	27.923
5-point Lobatto	2.326155917967643	.023570824973597	.0638	25.508
11-point Newton-Cote	2.304925264954663	.002340171960618	.1032	25.496
9-point Newton-Cotes	2.309464383406782	.006879290412736	.0849	25.476
4-point Chebyshev	2.255234587073967	.047350505920079	.0522	25.386
7-point Newton-Cotes	2.324715568045400	.022130475051354	.0664	24.932
16-point Newton-Cote	2.302836691040503	.000251598046457	.1496	24.067
10-point Newton-Cote	2.307779655612245	.005194562618199	.0970	23.549
12-point Newton-Cote	2.304389575052643	.001804482058597	.1166	23.540
5-point Chebyshev	2.270694358126710	.031890734867336	.0646	23.163
8-point Newton-Cotes	2.318756672902205	.016171579908159	.0780	22.964
20-point Newton-Cote	2.302624627351999	.000039534357953	.1923	22.896
30-point Newton-Cote	2.302585604701296	.000000511707250	.2792	22.535
2-point Gauss	2.106382978723405	.196202114270641	.0327	21.662
3-point Chebyshev	2.185206098280966	.117378994713080	.0434	21.441
6-point Newton-Cotes	2.359816280203757	.057231187209711	.0589	21.089
40-point Newton-Cote	2.302585101885737	.000000008891691	.3846	20.933
4-point Lobatto	2.399427496016199	.096842403022153	.0511	19.825
5-point Newton-Cotes	2.385700428603654	.083115335609608	.0629	17.177
10*5-point Gauss	2.302585083047175	.000000009946871	.4808	16.645
96-point Gauss	2.302585092994049	.000000000000004	.9191	15.721
50-point Newton-Cote	2.302585386278535	.000000293284489	.4630	14.111
9-point Romberg	2.313627920068950	.011042827074904	.1683	11.625
5-point Romberg	2.385700428603654	.083115335609608	.0948	11.394
4-point Newton-Cotes	2.563392857142857	.260807764148811	.0520	11.221
20*5-point Gauss	2.302585092966214	.000000000027832	.9585	11.013
5-point Simpson	2.407900969997744	.105315877003698	.0932	10.485
10-point Simpson	2.320575735326238	.017990642332192	.1689	10.330
17-point Romberg	2.303414977334841	.000829884340795	.3155	9.767
3-point Lobatto	2.740909090909091	.438323997915045	.0430	8.339
20-point Simpson	2.303996791535492	.001411698541446	.3539	8.054
33-point Romberg	2.302615169490732	.000030076496687	.6054	7.469
3-point Simpson	2.740909090909091	.438323997915045	.0549	6.523
3-point Romberg	2.740909090909091	.438323997915045	.0563	6.362
10-point trapezoidal	2.378968253968254	.076383160974208	.1879	5.946
65-point Romberg	2.302585558689706	.000000465695660	1.1882	5.329
5-point trapezoidal	2.629221182043763	.326636089049717	.0934	5.203
20-point trapezoidal	2.320713727992899	.018128634998853	.3731	4.668
60-point Newton-Cote	2.299587560505099	.002997532488947	.5739	4.397
129-point Romberg	2.302585095812346	.000000002818300	2.3372	3.658
100-point Simpson	2.302587417884465	.000002324890419	1.8364	3.068
100*5-point Gauss	2.302585092992895	.000000000001151	4.7955	2.490
257-point Romberg	2.302585093000290	.000000000006244	4.6682	2.400
100-point trapezoida	2.303266344275725	.000681251281679	1.8611	1.702
513-point Romberg	2.302585092994049	.000000000000004	9.2909	1.555
1025-point Romberg	2.302585092994044	.000000000000002	18.5000	.797
1000-point Simpson	2.302585093214436	.000000000220390	18.5833	.520
2049-point Romberg	2.302585092994055	.000000000000009	36.9667	.380
1000-point trapezoid	2.302591788824242	.000006695830197	18.5000	.280
4097-point Romberg	2.302585092994049	.000000000000003	73.8500	.196
8193-point Romberg	2.302585092994024	.000000000000022	148.3000	.092
10000-point Simpson	2.302585092994053	.000000000000007	185.6000	.076
3-point trapezoidal	3.293181818181818	.990596725187773	.0556	.074
16385-point Romberg	2.302585092994025	.000000000000020	295.5000	.046
10000-point trapezoi	2.302585159832371	.000000066838326	185.7000	.039