Tennessee Valley Authority
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# DEVELOPMENT OF THE FINITE-INTEGRAL METHOD

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## ABSTRACT

The relationships governing fields, stress and strain, hydrodynamics, and thermodynamics are frequently expressed as integral or differential equations. Since the introduction of digital computers many methods have been developed for translating these expressions of integral and differential calculus into algebraic expressions, which may be solved by numerical means. The two methods most commonly applied are finite-difference and finite-element. These methods are based on differential and variational calculus, respectively. A third and less commonly used method, the finite-integral method based on integral calculus, is presented. The finite-integral method is compared to the finite-difference and finite-element methods, and some of the relative advantages and disadvantages of each are cited. Specific examples are given for some common engineering applications. Particular emphasis is placed on the application of the finite-integral method to the Reynolds Transport Theorem.

## INTRODUCTION

In the course of solving integral and differential equations by digital means it is necessary to translate the expressions of calculus into expressions of algebra. Specifically, it is necessary to approximate integrations and differentiations by sums and differences. The two methods most commonly applied are the finite-difference method (FDM) and the finite-element method (FEM). The purpose herein is to introduce a third and less commonly used method: the finite-integral method (FIM). Each method has relative advantages and disadvantages which make one method preferable over another for a particular application. It is necessary to understand the strengths and weaknesses of each to determine which is best suited to an application.

## MATHEMATICAL BASIS

Understanding the mathematical basis for the finite-difference, finite-element, and finite-integral methods is the key to understanding the differences between the three. The FDM is based on differential calculus, the FEM is based on variational calculus, and the FIM is based on integral calculus. Before a problem can be solved by one of these three methods it must be cast in the corresponding form (differential, variational, or integral) by analytical means. The problem statement should be mathematically equivalent in each case. However, the three methods begin to diverge when approximations are made (e.g., truncated series expansions and assumed distributions to facilitate integration). The key assumption here is that the character of the original problem can be substantially preserved when making such approximations. The three methods also diverge in the process of translating the approximate analytical relationships into digital expressions.

## COMPUTATIONAL CELLS

The first concept which must be developed in describing a digital/numerical algorithm to solve integral or differential equations

is the cell. The region of interest must be approximated by a finite number of discrete subregions (cells) because of the finite capacity of digital computers. The resolution of calculus is infinite; whereas, the resolution of digital computers is finite in precision, storage, and speed. All three methods of translating the infinite resolution of calculus into finite resolution numerical algorithms approximate the region by an ensemble of cells, each having an analytical description that is relatively simple compared to the original problem. This involves an implicit assumption called the ensemble hypothesis: the whole may be constructed by the ordered assemblage of distinguishable parts, none of which possesses all the characteristics of the whole. Applying the ensemble hypothesis is to assume that the character of the original problem can be substantially preserved by solving a significantly simpler, yet similar, problem in each of several cells and assembling these without actually solving the original problem anywhere in the region.

The shape of the cells is an important consideration in selecting a method for solving integral or differential equations. Irregular shaped regions are more easily approximated by triangles in two dimensions, and tetrahedrons in three dimensions, than rectangular parallelepipeds. The FEM is typically developed for triangles and the FDM is typically developed for rectangles, making the FEM more appealing than the FDM for applications involving irregular regions. The FIM may be developed for triangles, tetrahedrons, or parallelepipeds.

## COMPUTATIONAL NODES

A node is a specific location within the region. A node need not be stationary and its coordinates need not be physical space and time. Nodes represent the points of intersection between the finite resolution digital domain and the infinite resolution analytical domain.

The relationship between the cells and nodes is an important feature of a particular method. The two basic classifications of cell-node relationships are boundary nodes and internal nodes. Boundary

nodes lie on the boundary of the cell and are typically at the corners, thus corner-boundary nodes (CBN), or at the midpoint between corners, thus midpoint-boundary nodes (MBN). Internal nodes are typically at the center of each face of a cell, thus central-face nodes (CFN). These are illustrated in Figure 1.

The FEM typically uses corner-boundary nodes, whereas, the FDM typically uses central-face nodes. In the case where values on the boundary of a cell are needed, the FDM may use both central-face nodes and midpoint-boundary nodes. This is referred to as a staggered grid and is shown in Figure 2. For the FDM these values on the cell boundary are typically secondary quantities, interpolated from surrounding central-face values. In contrast, the FIM uses only midpoint-boundary nodes. In the case where values in the interior of a cell are required, the FIM may use an average of the midpoint-boundary values.

## THE CONTINUITY EQUATION AS AN EXAMPLE

The steady continuity equation from the Eulerian point of view in two-dimensional rectangular coordinates is expressed as follows: (Section 4.2 of White)

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (1)$$

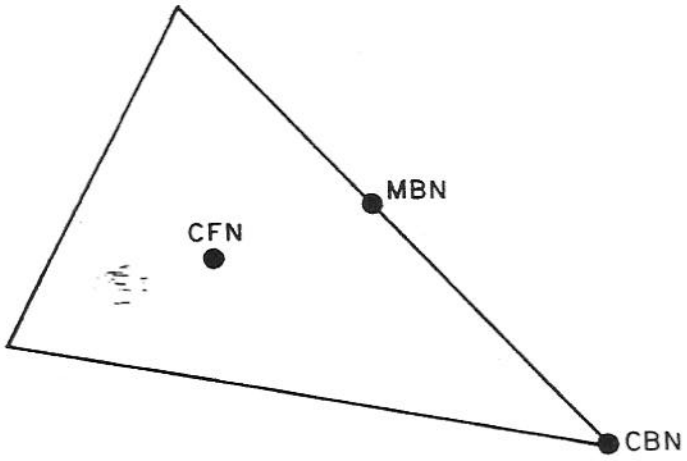
Where x and y are the two spatial directions, u and v are the components of velocity in the x and y directions respectively, and $\rho$ is the density.

### Finite-Difference Method

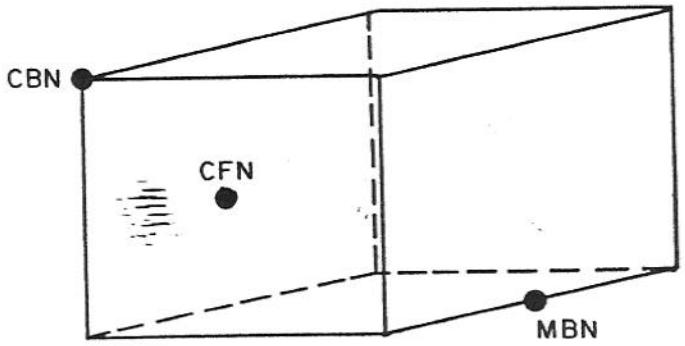
A FDM expression for Equation 1 would be as follows: (refer to Figure 3 for notation)

$$\frac{\rho(I, J+1)\, u(I, J+1) - \rho(I, J-1)\, u(I, J-1)}{x(J+1) - x(J-1)} + \frac{\rho(I-1, J)\, v(I-1, J) - \rho(I+1, J)\, v(I+1, J)}{y(I-1) - y(I+1)} = 0 \qquad (2)$$

Here standard matrix notation is used (viz. The 1,1 entry in a matrix is the upper left-hand corner. The first index, I, refers to the row number




TRIANGULAR CELL




RECTANGULAR PARALLELEPIPED CELL

Figure 1 : Relative Locations of Cell Boundaries and Nodes

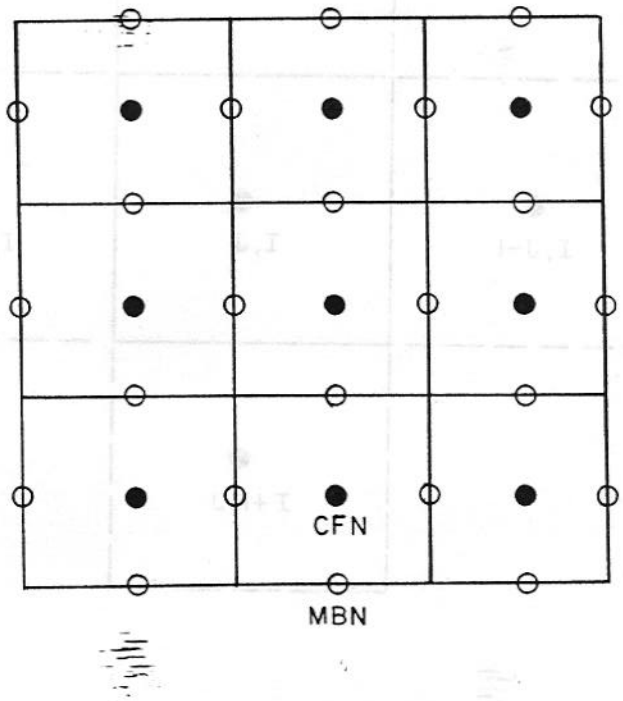



Figure 2: Staggered Rectangular Grid

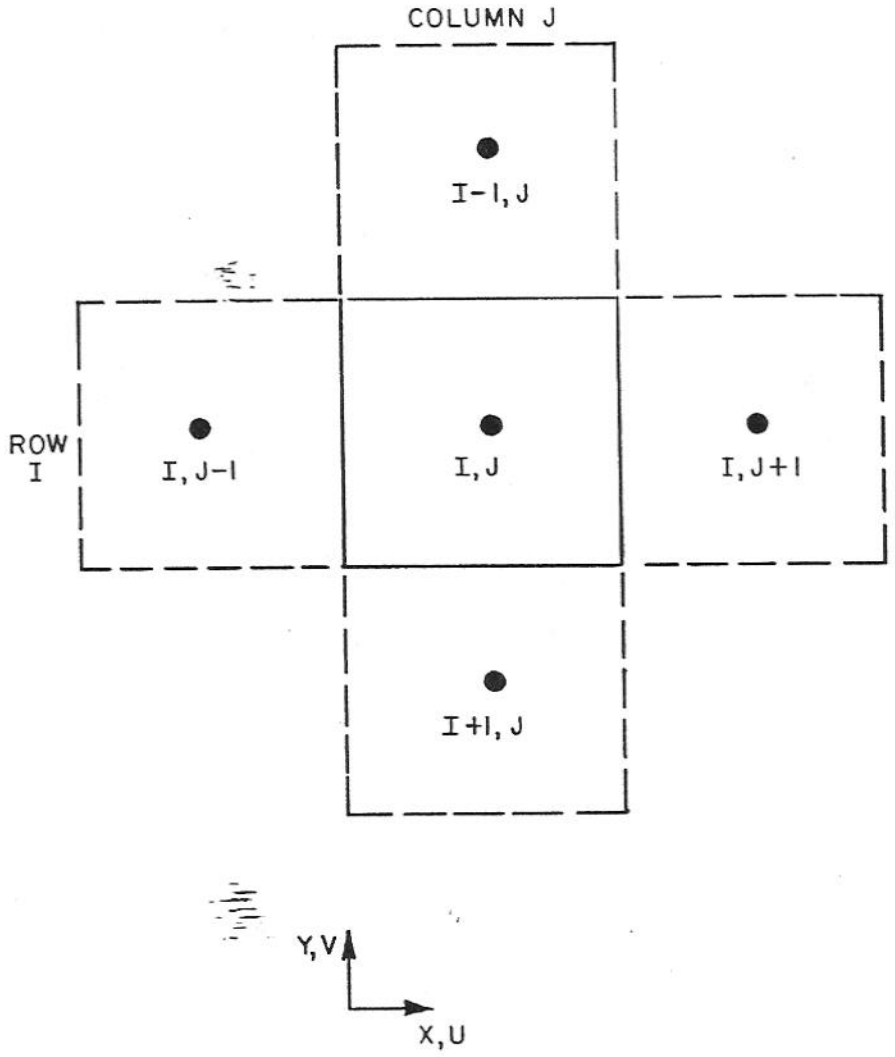



Figure 3: FDM Cell and Index Notation

and y location; and the second index, J, refers to the column number and x location).

Finite-Element Method

Using Galerkin's method as detailed in Section 9.5.3 of Huebner, the FEM variational statement may be derived for Equation 1 as follows. Equation 1 is integrated over the element (see Figure 4 for notation).

$$\Omega = \iint \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} \, dxdy \tag{3}$$

Applying Green's lemma (Section 13.4 of Wylie) transforms Equation 3 into an integral along the boundary.

$$\Omega = \int(\rho u dy - \rho v dx) \tag{4}$$

To perform the integral in Equation 4 it is typically assumed that $\rho u$ and $\rho v$ vary linearly along the boundaries. Thus the following expression is obtained for the integral.

$$\begin{aligned}\Omega = {} & \rho_1 u_1 (y_2 - y_3) + \rho_2 u_2 (y_3 - y_1) + \rho_3 u_3 (y_1 - y_2) \\ & + \rho_1 v_1 (x_3 - x_2) + \rho_2 v_2 (x_1 - x_3) + \rho_3 v_3 (x_1 - x_2)\end{aligned} \tag{5}$$

The variational statement is expressed for the ensemble by the following partial derivitives.

$$\frac{\partial \Sigma\Omega}{\partial u_i} = \frac{\partial \Sigma\Omega}{\partial v_i} = 0; \; i = 1, 2, 3 \tag{6}$$

Finite-Integral Method

The FIM is developed for the continuity equation by a different approximation for Equation 4. The FIM uses the midpoint-boundary node values to represent the average quantity along a boundary. For a triangular cell this may be expressed by the following (see Figure 5 for notation).

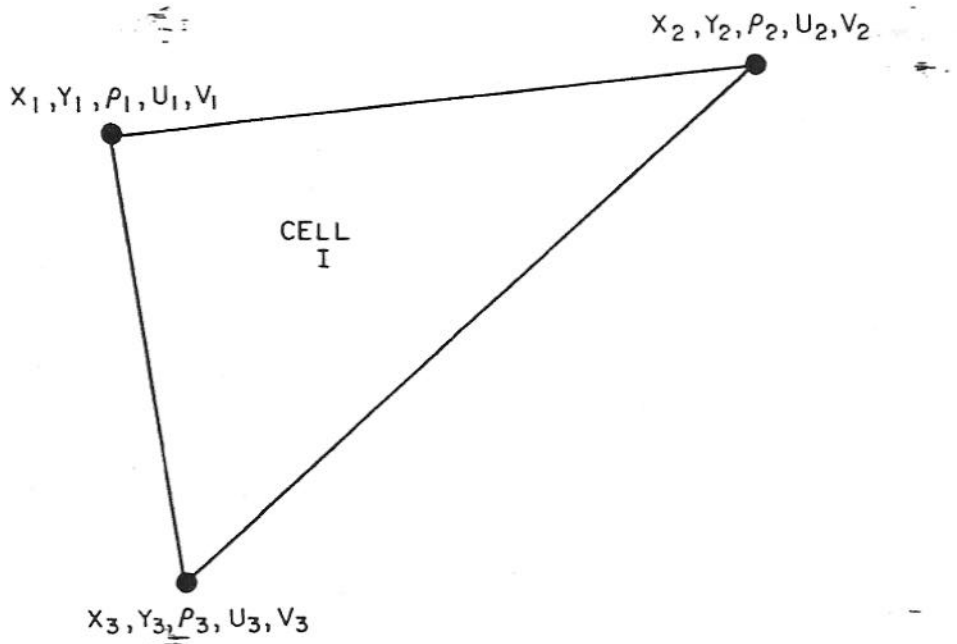



Figure 4 : FEM Cell and Index Notation

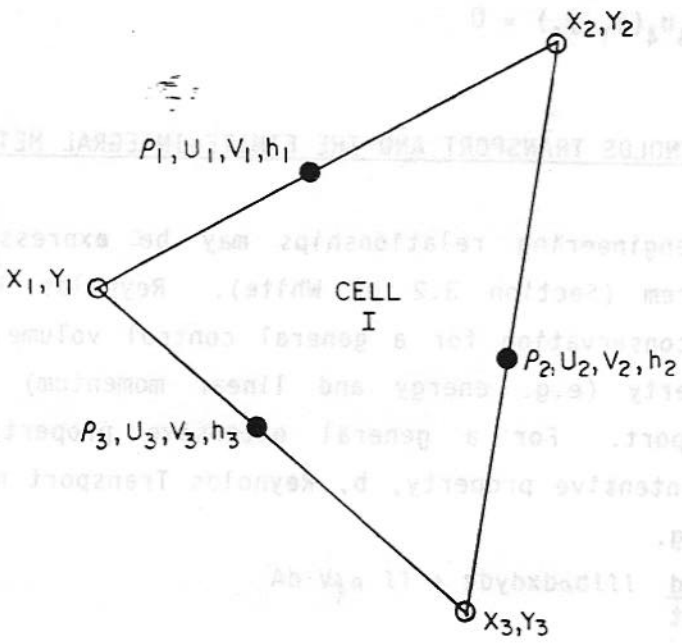



Figure 5 : Triangular FIM Cell and Index Notation

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$$\rho_1 u_1(y_2-y_1) + \rho_2 u_2(y_3-y_2) + \rho_3 u_3(y_1-y_3) \tag{7}$$

$$+\rho_1 v_1(x_1-x_2) + \rho_2 v_2(x_2-x_3) + \rho_3 v_3(x_3-x_1) = 0$$

Similarly the FIM may be developed for a rectangular element (see Figure 6 for notation).

$$\rho_1 v_1(x_1-x_2) + \rho_2 u_2(y_3-y_2) + \rho_3 v_3(x_3-x_4)$$

$$+ \rho_4 u_4(y_1-y_4) = 0 \tag{8}$$

## REYNOLDS TRANSPORT AND THE FINITE-INTEGRAL METHOD

Many engineering relationships may be expressed by Reynolds Transport Theorem (Section 3.2 of White). Reynolds Transport is an expression of conservation for a general control volume. That is, any conserved property (e.g. energy and linear momentum) is governed by Reynolds Transport. For a general extensive property, B, and its corresponding intensive property, b, Reynolds Transport may be expressed by the following.

$$\frac{dB}{dt} = \frac{d}{dt} \iiint b\rho dx dy dz + \iint \rho \vec{V} \cdot d\vec{A} \tag{9}$$

Where t is time; x, y, and z are spatial coordinates; $\rho$ is density; and $\vec{V} \cdot d\vec{A}$ is the dot product of the vector velocity, $\vec{V}$, and the differential outward-normal control surface area vector, $d\vec{A}$. B and b may be scalar or vector quantities. When using Reynolds Transport with the FIM, each cell is considered to be a separate control volume, and the boundary of the cell is the control surface.

The previous example of the continuity equation may be obtained from Reynolds Transport by setting b = 1. The term dB/dt becomes the rate of mass generation, which is zero. For steady conditions the first integral is also zero, leaving

$$0 = \iint \rho \vec{V} \cdot d\vec{A} \tag{10}$$

which is expressed algebraically as before.

The conservation of linear momentum is given by Newton's Second Law

$$g_c F = \frac{d(mV)}{dt} \tag{11}$$

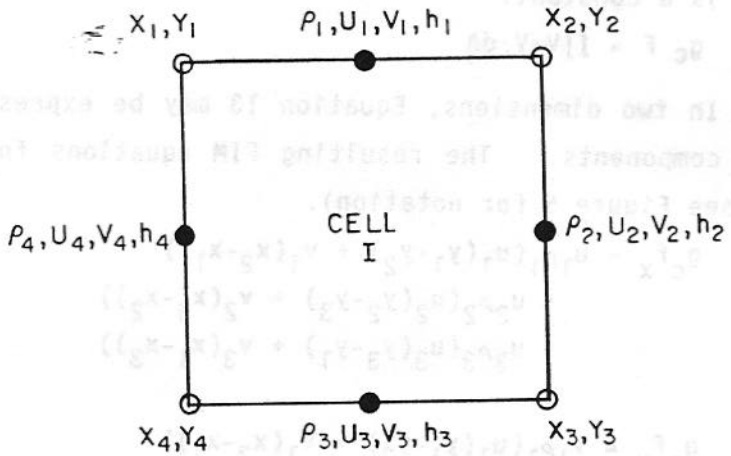


Figure 6 : Rectangular FIM Cell and Index Notation

where $g_c$ is Newton's constant, F is the vector force on the control volume, m is the mass of the control volume, V is the vector velocity, and t is time. If B=mV and b=V in Reynolds Transport, then Equations 9 and 11 may be combined to yield

$$g_c F = \frac{d}{dt} \iiint \rho V dx dy dz + \iint V \rho V \cdot dA \tag{12}$$

If the flow is steady and the control volume is fixed in space the first integral is a constant.

$$g_c F = \iint V \rho V \cdot dA \tag{13}$$

In two dimensions, Equation 13 may be expressed in terms of its x and y components. The resulting FIM equations for a triangular cell become (see Figure 5 for notation).

$$\begin{aligned} g_c f_x = {} & u_1 \rho_1 (u_1 (y_1 - y_2) + v_1 (x_2 - x_1)) \\ & + u_2 \rho_2 (u_2 (y_2 - y_3) + v_2 (x_3 - x_2)) \\ & + u_3 \rho_3 (u_3 (y_3 - y_1) + v_3 (x_1 - x_3)) \end{aligned} \tag{14}$$

$$\begin{aligned} g_c f_y = {} & v_1 \rho_1 (u_1 (y_1 - y_2) + v_1 (x_2 - x_1)) \\ & + v_2 \rho_2 (u_2 (y_2 - y_3) + v_2 (x_3 - x_2)) \\ & + v_3 \rho_3 (u_3 (y_3 - y_1) + v_3 (x_1 - x_3)) \end{aligned} \tag{15}$$

where $f_x$ and $f_y$ are the x and y components of the force vector. For the rectangular cell (see Figure 6 for notation).

$$\begin{aligned} g_c f_x = {} & u_1 \rho_1 v_1 (x_2 - x_1) + u_2 \rho_2 u_2 (y_2 - y_3) + u_3 \rho_3 v_3 (x_y - x_3) \\ & + u_4 \rho_4 v_4 (y_4 - y_1) \end{aligned} \tag{16}$$

$$\begin{aligned} g_c f_y = {} & v_1 \rho_1 v_1 (x_2 - x_1) + v_2 \rho_2 u_2 (y_2 - y_3) + v_3 \rho_3 v_3 (x_4 - x_3) \\ & + v_4 \rho_4 v_4 (y_4 - y_1) \end{aligned} \tag{17}$$

Even with fixed cells and steady conditions the derivitive of the first integral in Equation 12 is not always zero. An example of this would be a steady chemical reaction. For instance, chemical species may react to form new species while liberating specific energy $\varepsilon$. The reaction rate per unit mass, R, may be a function of local variables but is independent of time. The conservation of energy may be expressed by Reynolds Transport.

$$\frac{dE}{dt} = \iiint \rho R_c \, dx\,dy\,dz + \iiint \rho h \mathbf{V} \cdot d\mathbf{A} = 0 \qquad (18)$$

where h is the local enthalpy and may depend on temperature, pressure, and composition. For a triangular cell (see Figure 5 for notation) the FIM equation would be:

$$R_c \, (\rho_1 + \rho_2 + \rho_3)((x_3 - x)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_2))/6$$
$$+ \rho_1 h_1 (u_1 (y_1 - y_2) + v_1 (x_2 - x_1)) + \rho_2 h_2 (u_2 (y_3 - y) + v_2 (x_3 - x_2))$$
$$+ \rho_3 h_3 (u_3 (y_3 - y_1) + v_3 (x_1 - x_3)) = 0 \qquad (19)$$

For a rectangular cell (see Figure 6 for notation) the FIM equation would be

$$R_c (\rho_1 + \rho_2 + \rho_3 + \rho_4) \; (x_2 - x_1)(y_2 - y_3)/4$$
$$+ \rho_1 h_1 v_1 (x_2 - x_1) + \rho_2 h_2 u_2 (y_2 - y_3) + \rho_3 h_3 v_3 (x_4 - x_3)$$
$$+ \rho_4 h_4 u_4 (y_4 - y_1) \quad = 0 \qquad (20)$$

## BOUNDARY CONDITIONS

Zeroth order (constant) boundary conditions are applied in the same manner for the FDM, FEM, or FIM.

Many first order boundary conditions may be expressed in terms of the following general equation (see Figure 5 for notation).

$$a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} = b_0 + b_1 u_1 + b_2 u_2 + b_3 u_3 \qquad (21)$$

For a FIM triangular cell it may be assumed that the distribution of u is linear in x and y. The partial derivitives may be obtained algebraically from the midpoint-boundary values and the (x,y) location for the corners of the cell. The following equations may be obtained by algebraic manipulation of the assumed linear distribution with a midpoint-boundary node triangular cell (similar relationships may be derived for a rectangular cell).

$$\frac{\partial u}{\partial x} = C_1 u_1 + C_2 u_2 + C_3 u_3 \qquad (22)$$

$$\frac{\partial u}{\partial y} = C_4 u_1 + C_5 u_2 + C_6 u_3 \qquad (23)$$

$$C_1 = (y_2 - y_1)/2D \tag{24}$$

$$C_2 = (y_3 - y_2)/2D \tag{25}$$

$$C_3 = (y_1 - y_3)/2D \tag{26}$$

$$C_4 = (x_1 - x_2)/2D \tag{27}$$

$$C_5 = (x_2 - x_3)/2D \tag{28}$$

$$C_6 = (x_3 - x_1)/2D \tag{29}$$

$$D = (x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2)/4 \tag{30}$$

Equations 21 - 30 may be used to translate first order boundary conditions into algebraic expressions. These then become additional simultaneous equations to be added to the ensemble set.

## BUILDING THE ENSEMBLE

With the FDM each governing equation for each cell, together with the boundary conditions, is an independent equation. The FDM ensemble is built from the simultaneous set of these independent equations. The equations obtained for a single cell or boundary condition, however, are not independent with the FEM. The FEM ensemble is obtained by accumulating the gradient of the variational equation for all of the cells and boundary conditions. The FIM ensemble is obtained in the same manner as the FDM. With the FDM and FIM, an independent equation is obtained as each cell or boundary condition is considered, whereas, with the FEM all adjacent cells must be considered to obtain a single independent equation. This distinction between the FDM, FIM, and FEM impacts the storage requirements and the solution technique. The FEM equations cannot be solved cell-by-cell as in the case of the FDM and FIM. The FEM requires concurrent storage of the entire set of simultaneous equations representing the ensemble, whereas, the FDM and FIM do not. When effecting an iterative solution, the FDM and FIM equations may be computed cell-by-cell as needed. Other than reduced storage this cell-by-cell solution with current update values may have

significant advantage over methods where the set of equations is solved before updating (see Section 13.1 of Ortega and Rheinboldt).

## RELATIVE ADVANTAGES AND DISADVANTAGES OF THE METHODS

The FDM is typically the easiest of the three to develop. The FIM is relatively simple to develop for certain cases such as when Reynolds Transport is applicable. The FIM may require the use of Green's Theorem or Stokes' Theorem, making it more involved than the FDM. The FEM is frequently taxing to the engineer with a strong analytical background.

The FEM and FIM may be developed for triangular and tetrahedral cells making these methods more attractive for applications having irregular regions.

The FDM uses internal nodes, whereas, the FEM and FIM use boundary nodes making the FEM and FIM more attractive in terms of describing some boundary conditions.

The FIM uses midpoint nodes, which avoids the problem of conflicting boundary conditions at corners encountered with the FEM.

When solving a second order equation, the FDM has a bandwidth of five, the FEM has a bandwidth dependent on the number of nearest neighbors (typically 8), and the FIM has a bandwidth of four making the FIM the most advantageous.

The FEM requires concurrent storage of the entire set of simultaneous equations in order to effect a solution. The FDM and FIM do not require this concurrent storage and may be solved by the Gauss-Seidel (point-by-point successive substitution) or similar point-by-point methods (see Chapter 7 of Ortega and Rheinboldt) resulting in a significant reduction of storage required compared to the FEM and some advantage with nonlinear problems by virtue of current updating. The FEM may also be solved by the Gauss-Seidel method, but without updating the nonlinearities.

## CONCLUSIONS

The finite-integral method can be used to translate integral equations into algebraic expressions in order to effect a numerical solution. The FIM can be developed for several different shaped cells, making it an attractive alternative to the finite-difference method for irregular shaped regions. The FIM may be solved using current update (point-by-point) methods, making it an attractive alternative to the finite-element method for nonlinear problems and cases where storage is limited. The FIM is relatively simple to develop for problems where Reynolds Transport is applicable. The FIM should be added to the list of mathematical tools at the disposal of engineers.

## NOMENCLATURE

| | |
|---|---|
| $A$ | outward-normal control surface area vector |
| $B$ | general extensive property |
| $b$ | general intensive property |
| $E$ | energy |
| $F$ | force vector |
| $f_x$ | x component of F |
| $f_y$ | y component of F |
| $g_c$ | Newton's constant |
| $h$ | enthalpy |
| $I$ | row index |
| $J$ | column index |
| $m$ | mass |
| $R$ | reaction rate per unit mass |
| $t$ | time |
| $u$ | x component of V |
| $v$ | y component of V |
| $V$ | velocity vector |
| $x$ | first spatial direction |
| $y$ | second spatial direction |
| $z$ | third spatial direction |
| $\varepsilon$ | specific reaction energy |
| $\rho$ | density |
| $\Omega$ | variational integral |

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